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# A phase-equation approach to boundary-layer instability theory: Tollmien–Schlichting waves

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Our concern is with the evolution of large-amplitude Tollmien–Schlichting waves in boundary-layer flows. In fact, the disturbances we consider are of a comparable size to the unperturbed state. We shall describe two-dimensional disturbances which are locally periodic in time and space. This is achieved using a phase equation approach of the type discussed by Howard & Kopell (1977) in the context of reaction–diffusion equations. We shall consider both large and $O(1)$ Reynolds number flows though, in order to keep our asymptotics respectable, our finite-Reynolds-number calculation will be carried out for the asymptotic suction flow. Our large-Reynolds-number analysis, though carried out for Blasius flow, is valid for any steady two-dimensional boundary layer. In both cases the phase-equation approach shows that the wavenumber and frequency will develop shocks or other discontinuities as the disturbance evolves. As a special case we consider the evolution of constant frequency/wavenumber disturbances and show that their modulational instability is controlled by Burgers equation at finite-Reynolds-number and by a new integro-differential evolution equation at large-Reynolds-numbers. For the large Reynolds number case the evolution equation points to the development of a spatially localized singularity at a finite time.

## 1. Introduction

Most boundary layers of practical importance are susceptible to a variety of instability mechanisms which lead to the onset of transition to turbulence. Usually, more than one mechanism will be operational in any particular case, and a full understanding of how transition occurs will require a detailed understanding of the nonlinear interaction of the different modes of instability. Here, we will concern ourselves with the strongly nonlinear evolution of a slowly varying Tollmien–Schlichting wave system. In the first instance we consider lower branch Tollmien–Schlichting waves which are known to be governed by triple-deck theory (e.g. Smith 1979*a*,*b*; Hall & Smith 1984; Smith & Burggraf 1985). Then we shall consider the corresponding problem at finite Reynolds numbers. It is worth pointing out that the approach we use here can be used to describe other modes of instability and is, in fact, based on ideas given some years ago by Howard & Kopell (1977) who were interested in the evolution of nonlinear wave systems in reaction–diffusion equations.

The first application of triple-deck theory to describe the linear and nonlinear growth of lower branch Tollmien–Schlichting waves is apparently due to Smith (1979*a*,*b*) though Lin (1966) clearly recognized the appropriate large-Reynolds-number scalings for Tollmien–Schlichting waves long before triple-deck theory was invented. The investigation of Smith (1979*a*) showed how non-parallel effects could be taken care of in a self-consistent manner using asymptotic methods. Previously Gaster (1974) used a successive approximation procedure to tackle the same kind of problem. Subsequently Smith (1979*b*) showed how the nonlinear growth of Tollmien–Schlichting waves could be taken care of using triple-deck theory. However, the results of Smith (1979*b*), and the subsequent extension to three-dimensional modes by Hall & Smith (1984), are confined to the weakly nonlinear stage where an ordinary differential amplitude equation describes the initial stage of the bifurcation from a linearly growing disturbance. Some years later Smith & Burggraf (1985) discussed the high-frequency limit of the lower branch triple-deck problem and uncovered a sequence of nonlinear structures governing a sequence of successively more nonlinear wave interactions. Related work on the high-frequency limit had been previously carried out by Zhuk & Ryzhov (1982).

Subsequently Smith & Stewart (1987) investigated the interaction of three-dimensional modes at high frequencies and obtained reasonable agreement with the experiments of Kachanov & Levchenko (1984). However, a referee of this paper pointed out to the author that in a recent paper Khokhlov (1994) claims that the work of Smith & Stewart (1987) is in error because of its incorrect treatment of the critical layer.

In the first instance we shall restrict our attention to two-dimensional waves and determine how the wavenumber and frequency of a nonlinear wave system may be found as it moves through a growing boundary layer. This problem has not yet been addressed. Intuitively one would expect that a small-amplitude wave would evolve from its weakly nonlinear form into a larger amplitude state until it is described by the Smith–Burggraf structure at sufficiently large values of the local frequency of the disturbance. Our calculations suggest that this is not the case since we were unable to find finite-amplitude periodic solutions at large frequencies. However, it could be that finite-amplitude periodic states exist at high frequencies but are not continuously connected with the bifurcation point of linear theory.

The asymptotic structure we use is based on the so-called 'phase-equation' approach used so successfully to describe large-amplitude Bénard convection in large containers by, amongst others, Kramer, Ben Jacob, Brand & Cross (1982); Cross & Newell (1984); Newell, Passot & Lega (1993). Using this approach it has been possible to describe the experimentally observed slowly varying planform of Bénard convection. Thus, for example, the dislocation of convection rolls is now reasonably well understood using the phase equation approach. Interestingly enough, it turns out that the essential ideas of this approach had been elucidated in the context of travelling wave instabilities several years earlier, see Howard & Kopell (1977) and indeed the method can be found in Whitham (1974). The evolution of travelling waves in a Blasius boundary layer is the subject of the first part of this work and not surprisingly the analysis to be used has similarities with that of the latter authors.

The essential idea behind the phase-equation approach may be explained in the following manner. Suppose there exists some flow which is unstable to a travelling wave disturbance of wavenumber $\alpha$ and frequency $\Omega$. For a fully nonlinear disturbance the frequency $\Omega$ will be a function of $\alpha$ which itself can be thought of as a function of $\Delta$, a measure of the size of disturbance. If we let $\Delta$ tend to zero then, for small $\Delta$, the quantities $\alpha$ and $\Omega$ will differ from their linear neutral values by $O(\Delta)^2$ so that finite-amplitude disturbances begin as supercritical bifurcations from the basic state. For $O(1)$ values of $\Delta$ the quantities $\alpha$ and $\Omega$ are accessible only by numerical means, see for example Herbert (1977) for details of the computation of $\alpha$ and $\Omega$

for Tollmien–Schlichting waves in plane Poiseuille flow or Conlisk, Burggraf & Smith (1987) for a similar calculation for Tollmien–Schlichting waves in Blasius boundary layers at large Reynolds numbers. In some cases the frequency of the waves is zero and the wavenumber of the disturbances may be sensibly held fixed when the control parameter or disturbance size is varied, see for example Hall (1988) for a discussion of the fully nonlinear Görtler problem in a growing boundary layer. For a travelling wave disturbance in a growing boundary layer we expect that the wavenumber and frequency of the disturbance should change as it propagates into locally less or more unstable parts of the flow. The phase-equation approach provides a rational framework for following such an evolution. If the wave has local frequency and wavenumber which are $O(1)$ with respect to the variables $t$ and $x$ we introduce slowly varying variables $T$ and $X$ by writing

$$T = \delta t, \quad X = \delta x,$$

and we now think of $\alpha$ and $\Omega$ as being functions of $X$ and $T$. Thus we introduce the phase function $\Theta(X, T)$ defined by $\Theta = \theta(X, T)/\delta$ with

$$\alpha = \theta_X, \quad \Omega = -\theta_T$$

and as a consequence the wave system evolves such that

$$\alpha_T + \Omega_X = 0. \tag{1.1}$$

Partial derivatives with respect to $x$ and $t$ must then be replaced using

$$\frac{\partial}{\partial x} \to \alpha \frac{\partial}{\partial \Theta} + \delta \frac{\partial}{\partial X}, \quad \frac{\partial}{\partial t} \to -\Omega \frac{\partial}{\partial \Theta} + \delta \frac{\partial}{\partial T},$$

and if we then equate terms of $O(\delta)^0$ we recover the unmodulated equations of motion with $\alpha$ and $\Omega$ playing the role of wavenumber and frequency. Thus the leading-order problem using the phase equation approach is simply the unmodulated case with $\alpha$ and $\Omega$ being functionally related in order that the system, with a given disturbance size, has a solution. At next order (usually $O(\delta)$ but in fact $O(\delta^{1/3})$ in triple-deck problems) we obtain a linearized inhomogeneous form of the leading-order problem. Owing to the invariance of the problem under a translation in the $x$-direction it is easy to see that the linearized homogeneous form of this system has a non-trivial solution so that some solvability condition must be satisfied if the inhomogeneous problem is to have a solution. This solvability condition is satisfied by introducing an expansion of $\Omega$ in appropriate powers of $\delta$. The solution of this problem then enables us to write down the asymptotic form of (1.1) up to the second order. This procedure can be continued in principle to any order and the coefficients in the expansion of $\Omega$ are found as solvability conditions at each order. The evolution of a given wave system can then be found by the solution of the calculated asymptotic approximation to (1.1). We find that (1.1) takes on a particularly simple form if the wave system has fixed wavenumber and frequency at leading order. We shall in this paper calculate (1.1) correct up to second order for both wave systems governed by triple-deck theory and those satisfying the two-dimensional Navier–Stokes equations. We can show that at high Reynolds numbers (1.1), may then be reduced to the form

$$\frac{\partial \Lambda}{\partial \tau} + \Lambda \frac{\partial \Lambda}{\partial \xi} = \frac{\partial}{\partial \xi} \int_{-\infty}^{\infty} \frac{\partial \Lambda(s)}{\partial s} \beta(\xi - s) \mathrm{d}s, \tag{1.2}$$

where $\Lambda$ is a wavenumber perturbation and $\beta$ will be defined later.

This is in effect the evolution equation for a wave packet of large-amplitude Tollmien–Schlichting waves in a growing boundary layer.

At finite Reynolds numbers the modulation equation corresponding to (1.2) is

$$A_\tau + AA_\xi = \pm A_{\xi\xi}. \tag{1.3}$$

Thus at finite Reynolds numbers Burgers equations controls the slow dynamics of a two-dimensional wave system.

The procedure adopted in the rest of this paper is as follows: in §2 we derive the phase equation for two-dimensional triple-deck problems. In §3 we describe the numerical work required to determine the quantities appearing in the equation. In §4 we look at the special case of almost uniform wavetrains and derive (1.2). In §5 we show how the equivalent of (1.2) can be derived for amplitude perturbations by a more conventional multiple-scale approach directly from the triple-deck equations. The phase-equation approach for a boundary layer at finite Reynolds numbers is then discussed in §6. The modulation equation (1.3) is derived in that section as a special case for almost uniform wavetrains. Finally in §7 we draw some conclusions and give the generalized form of the evolution equations which accounts for non-parallel effects.

## 2. Derivation of the phase equation for two-dimensional triple-deck problems

Our concern is with the structure of fully nonlinear solutions of the triple-deck equations governing the evolution of two-dimensional Tollmien–Schlichting waves in incompressible boundary layers. Following the usual notation, e.g. Smith (1979*a*), the appropriate differential equations in scaled form are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1a}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2}, \tag{2.1b}$$

which must be solved subject to

$$u = v = 0, \quad y = 0, \tag{2.2a}$$

$$u \sim y + A(x,t), \quad y \to \infty, \tag{2.2b}$$

$$p = \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{\partial A}{\partial s} \frac{\partial s}{(x-s)} \mathrm{d}s. \tag{2.2c}$$

The displacement function $A$ and the pressure $p$ depend only on $x$ and $t$ and if we wish to consider other flows the pressure displacement law (2.2*c*) must be modified accordingly. Linear Tollmien–Schlichting waves correspond to perturbing $u$ in the form

$$u = y + U(y)\mathrm{e}^{\mathrm{i}\alpha\{x-\Omega t\,\alpha\}}, \tag{2.3}$$

with $U$ small and, from Smith (1979*a*), the eigenrelation takes the form

$$\mathrm{Ai}'(\xi_0) = (\mathrm{i}\alpha)^{1/3}\alpha \int_{\xi_0}^{\infty} \mathrm{Ai}(\eta)\mathrm{d}\eta. \tag{2.4}$$

where Ai is the Airy function and $\xi_0 = -\mathrm{i}\Omega\,(\mathrm{i}\alpha)^{2/3}$. Solutions of (2.4) with $\Omega$ complex and $\alpha$ real show that neutral stability occurs for $\Omega \simeq 2.298$, $\alpha \simeq 1.001$, and that $\Omega_i$ is positive for all frequencies greater than the neutral value. In the high-frequency limit it can be shown from (2.4) that

$$\alpha = \Omega + O(\Omega^{-1/2}). \tag{2.5}$$

The above limit was discussed in detail by Smith & Buggraff (1985) who investigated the possible nonlinear structures which emerge in that limit, see also Zhuk & Ryzhov (1982). The structures found by Smith & Burggraf depend crucially on the fact that the right-hand side of (2.5) is complex only at order $\Omega^{-1/2}$ so that, even though a wave is never neutral, its small growth can be balanced at higher order by nonlinear effects. Here our interest is with the case $\alpha = O(1), \Omega = O(1)$, but we shall allow for a slow evolution of the wave system as it moves through the boundary layer. The essential details of our approach are to be found in Howard & Kopell (1977) who were concerned with slowly varying waves in reaction diffusion systems. As a first step we introduce slow time and space variables, $T$ and $X$, by writing

$$T = \delta t, \tag{2.6a}$$

$$X = \delta t, \tag{2.6b}$$

where $\delta$ is a small positive parameter. Note that at this stage $\delta$ is simply a device for introducing a modulation length into the problem; later it will be related to a wavenumber perturbation. We shall investigate the evolution of a fully nonlinear wavelike solution of (2.1)–(2.2), but allow the wavenumber and frequency to be slow functions of $X$ and $T$.

In order to describe such a structure we introduce a phase function $\theta(x,t)$ such that the wavenumber and frequency of the wave are defined by

$$\alpha = \frac{\partial\theta}{\partial x}, \quad \Omega = -\frac{\partial\theta}{\partial t}. \tag{2.7}$$

The wavenumber and frequency must therefore satisfy

$$\frac{\partial\alpha}{\partial t} + \frac{\partial\Omega}{\partial x} = 0, \tag{2.8}$$

and (2.8) therefore corresponds to the conservation of phase. Now we shall assume, following Howard & Kopell (1977), that $\alpha$ and $\Omega$ are functions only of $X$ and $T$. In that case (2.8) reduces to

$$\frac{\partial\alpha}{\partial T} + \frac{\partial\Omega}{\partial X} = 0. \tag{2.9}$$

and it is then convenient to write to phase variable $\Theta = \delta^{-1}\theta(X,T)$. The $x$ and $t$ derivatives in (2.1)-(2.2) must then be transformed according to

$$\frac{\partial}{\partial x} \to \alpha\frac{\partial}{\partial\Theta} + \delta\frac{\partial}{\partial X}, \tag{2.10a}$$

$$\frac{\partial}{\partial t} \to -\Omega\frac{\partial}{\partial\Theta} + \delta\frac{\partial}{\partial T}. \tag{2.10b}$$

We seek a locally wavelike solution of (2.1)–(2.2) and impose periodicity in the phase variable $\Theta$. It remains now for us to find a small $\delta$ solution of the full triple-deck problem (2.1)–(2.2). At first sight, in view of (2.10), we would expect to develop a solution of that system in terms of $\delta$. However, it turns out that for $\delta << 1$ the leading-order approximation to (2.1)–(2.2) has a mean term correction which depends on the slow variable $X$. This mean flow is modified in an outer $O(\delta^{-1/3})$ diffusion layer. The expansions must therefore proceed in powers of $\delta^{1/3}$ and we therefore write

$$\Omega = \Omega_0 + \delta^{1/3}\Omega_1 + \cdots, \tag{2.11a}$$
$$u = u_0(X,T,\Theta,Y) + \delta^{1/3}u_1(X,T,\Theta,Y) + \cdots, \tag{2.11b}$$
$$v = v_0(X,T,\Theta,Y) + \delta^{1/3}v_1(X,T,\Theta,Y) + \cdots, \tag{2.11c}$$
$$p = \delta^{-1/3}p_M(X,T) + p_0(X,T,\Theta) + \delta^{1/3}p_1(X,T,\Theta) + \cdots, \tag{2.11d}$$
$$A = A_0(X,T,\Theta) + \delta^{1/3}A_1(X,T,\Theta) + \cdots. \tag{2.11e}$$

Here we have anticipated the presence of what can be thought of as a pressure eigenfunction, $p_M$, in the expansion of the pressure. The need for this function will become clear later when we investigate the outer diffusion layer in which the mean flow correction adjusts so as to match with the main deck solution. The leading-order problem is then found to be

$$\alpha\frac{\partial u_0}{\partial\Theta} + \frac{\partial v_0}{\partial y} = 0, \tag{2.12a}$$

$$-\Omega_0\frac{\partial u_0}{\partial\Theta} + \alpha u_0\frac{\partial u_0}{\partial\Theta} + v_0\frac{\partial u_0}{\partial y} = -\alpha\frac{\partial p_0}{\partial\Theta} + \frac{\partial^2 u_0}{\partial y^2}, \tag{2.12b}$$

$$u_0 = v_0 = 0, \ \ y = 0, \tag{2.12c}$$

$$u_0 \sim y + A_0(X,\Theta,T) \ \ y \to \infty, \tag{2.12d}$$

$$p_0 = \frac{\alpha}{\pi}\int_{-\infty}^{\infty}\frac{(\partial A/\partial\psi)\mathrm{d}\psi}{(\Theta-\psi)}. \tag{2.12e}$$

Hence the leading-order problem is obtained from the full two-dimensional problem by restricting attention to solutions in the form of travelling waves of local wavenumber $\alpha$ and frequency $\Omega_0$. This specifies a nonlinear eigenvalue problem

$$\Omega_0 = \Omega_0(\alpha), \tag{2.13}$$

which must be determined numerically. At this stage we assume that (2.13) and the corresponding nonlinear eigenfunctions $u_0, v_0, p_0$ and $A_0$ are known. We further note $A_0$ may be written in the form

$$A_0 = \bar{A}_0(X,T) + \tilde{A}_0(\Theta,X,T) \tag{2.14}$$

where $\tilde{A}_0$ has zero mean with respect to $\Theta$. In order to match the mean flow correction with the flow in the main deck we must investigate the outer boundary layer where convective effects on the slow streamwise lengthscale come into play. Before we investigate the outer boundary layer it is convenient to discuss the next order system in the $y = O(1)$ region. The equations to be satisfied are

$$\alpha\frac{\partial u_1}{\partial\Theta} + \frac{\partial v_1}{\partial y} = 0, \tag{2.15a}$$

$$-\Omega_0\frac{\partial u_1}{\partial\Theta} + \alpha\left\{u_0\frac{\partial u_1}{\partial\Theta} + u_1\frac{\partial u_0}{\partial\Theta}\right\} + v_0\frac{\partial u_1}{\partial y} + v_1\frac{\partial u_0}{\partial y} + \alpha\frac{\partial p_1}{\partial\Theta} - \frac{\partial^2 u_1}{\partial y^2} = \Omega_1\frac{\partial u_0}{\partial\Theta}, \tag{2.15b}$$

subject to

$$u_1 = v_1 = 0, \ \ y = 0, \tag{2.15c}$$

and

$$p_1 = \frac{\alpha}{\pi} \int_{-\infty}^{\infty} \frac{\partial A_1/\partial s}{(x-s)} \mathrm{d}s. \tag{2.15d}$$

In addition we require a condition involving $u_1$ at the edge of the boundary layer. On the basis of (2.2*b*) we might expect that, $u_1 \to A_1$, $y \to \infty$, is the appropriate condition. However, $A_1$ is essentially the displacement function in the main deck and $u_1$ is modified in the outer boundary layer in which the mean flow correction evolves. For completeness of the $O(\delta^{1/3})$ system we anticipate the form of the matching condition found from a consideration of the outer layer. The appropriate condition is

$$u_1 \to D_1 + E_1 y, \quad y \to \infty, \tag{2.16}$$

where $D_1$ is to be found in terms of $A_1$ by a consideration of the outer boundary layer and $E_1$ will be found in terms of the mean flow correction in the outer layer. For large values of $y$ we have $u \sim y$ so that the thickness of the outer layer is fixed by the balance

$$y\delta \frac{\partial}{\partial X} \sim \frac{\partial^2}{\partial y^2}.$$

Hence we write

$$\eta = \delta^{1/3} y$$

and now develop an asymptotic solution of (2.1)–(2.2) valid for $\eta = O(1)$. The solution here is similar to that in the main deck for two-dimensional triple-deck problems. We write

$$u = \left\{\delta^{-1/3}\eta + u_M(X, \eta, T) + \cdots\right\} + \left\{U_0 + \delta^{1/3} U_1 + \cdots\right\},$$

$$v = \left\{\delta^{2/3} v_M(X, \eta, T) + \cdots\right\} + \frac{1}{\delta^{1/3}} \left\{V_0 + \delta^{1/3} V_1 + \cdots\right\},$$

$$p = \delta^{-1/3} \left\{p_M + \cdots\right\} + P_0 + \delta^{1/3} P_1 + \cdots.$$

Here the terms in the first bracket of each expansion correspond to the mean flow. If we substitute the above expansions into (2.1)–(2.2), and make the appropriate changes of variables, then we find that the functions $U_0, U_1$ are given by

$$U_0 = \tilde{A}_0,$$

and

$$U_1 = A_1 + \tilde{A}_0 u_{M\eta},$$

where $A_1$ is the displacement function in the main deck solution. The functions $u_M$ and $v_M$ are found to satisfy

$$\frac{\partial^2 u_M}{\partial \eta^2} - \eta \frac{\partial u_M}{\partial X} - v_M = p_{MX},$$

$$\frac{\partial u_M}{\partial X} + \frac{\partial v_M}{\partial \eta} = 0,$$

which are to be solved subject to

$$u_M = \bar{A}_0, \quad v_M = 0, \quad \eta = 0, \quad u_M \to F, \quad \eta \to \infty.$$

where $F$ is a displacement function related to $p_M$ by the usual pressure displacement law. The solution of the above problem for the mean flow correction is most easily obtained using a Fourier transform with respect to $X$ and indeed the solution of the problem is given by Smith (1973). For the purposes of our calculation we need only the quantity $u_{M\eta}(0)$ which gives the jump in $U_1$ which occurs across the diffusion layer. Following Smith (1973) we find that

$$u_{M\eta} = \frac{3\mathrm{Ai}(0)\chi^{4/3}}{2\pi}\int_{-\infty}^{\infty}\frac{\partial \bar{A}_0(q,T)}{\partial q}\beta(X-q)\mathrm{d}q = J, \tag{2.17}$$

with $\beta$ defined by

$$\beta = \int_0^{\infty}\frac{s^{2/3}(3^{1/2}s^{4/3}-1)\mathrm{e}^{-\chi s q}\mathrm{d}s}{1-3^{1/2}s^{4/3}+s^{8/3}} \quad (q>0),$$

$$\beta = -2\int_0^{\infty}\frac{s^{2/3}\mathrm{e}^{\chi s q}\mathrm{d}s}{1+s^{8/3}} \quad (q<0),$$

where $\chi = -3\mathrm{Ai}'(0) = 0.8272\cdots$. We see that the $O(\delta^{1/3})$ correction to the wavelike parts of the expansions in the lower deck and the transition layer match if

$$D_1 = A_1 + J, \tag{2.18}$$

whilst the mean parts match if

$$E_1 = J, \tag{2.19}$$

where $J$ is defined by (2.17).

The quantity $\Omega_1$ is now determined as a solvability condition on the inhomogeneous system specified by (2.15), (2.16) and (2.19). Such a condition is required because of the translational invariance of the leading-order wave system with respect to $\Theta$. Therefore a solution of the homogeneous problem is found by setting $(u_1, v_1, p_1, A_1) = \partial\, \partial\Theta(u_0, v_0, p_0, A_0)$. In order to find the appropriate solvability condition it is convenient to define $\boldsymbol{Z} = (p_1, v_1, u_1 - Jy, u_{1y} - J)^T$. We then must determine the condition that the inhomogeneous system given below has a solution:

$$\frac{\partial \boldsymbol{Z}}{\partial y} = \boldsymbol{B}\boldsymbol{Z} + \boldsymbol{C}\frac{\partial \boldsymbol{Z}}{\partial \Theta} + \Omega_1\boldsymbol{F}_1 + \boldsymbol{G}_1,$$

$$\boldsymbol{B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & u_{0y} & \alpha u_{0\Theta} & v_0 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & \alpha u_0 - \Omega_0 & 0 \end{bmatrix},$$

$$\boldsymbol{F}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ u_{0\Theta} \end{bmatrix}, \quad \boldsymbol{G}_1 = J\begin{bmatrix} 0 \\ 0 \\ 0 \\ v_0 + \alpha y u_{0\Theta} \end{bmatrix},$$

subject to

$$u_1 = v_1 = 0, \quad y = 0,$$

$$u_1 \to A_1 + J, \quad y \to \infty,$$

$$p_1 = \frac{\alpha}{\pi}\int_{-\infty}^{\infty}\frac{\partial A_1/\partial s}{X - s_1}.$$

The system adjoint to the homogeneous form of the above problem is

$$-\frac{\partial \boldsymbol{J}}{\partial y} = \boldsymbol{D}\boldsymbol{J} - \boldsymbol{C}^T\frac{\partial \boldsymbol{J}}{\partial \Theta},$$

with

$$\boldsymbol{D} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_{0y} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & v_0 \end{bmatrix},$$

$$\boldsymbol{J} = (M, N, S, T)^T,$$

and subject to the conditions

$$M = T = 0, \ y = 0,$$

$$S \to 0, \ y \to \infty,$$

$$M \to M_\infty(\Theta), \quad T \to T_\infty y^{-2}, \quad y \to \infty \text{ with } T_\infty = \frac{-1}{\alpha\pi} \int_{-\infty}^{\infty} \frac{M}{(x-s)} \mathrm{d}s.$$

The condition that the problem for $(u_1, v_1, p_1, A_1)$ has a solution is then found to be

$$\Omega_1 = JK(\alpha), \tag{2.20}$$

with

$$K(\alpha) = \frac{-\int_0^{2\pi} M_\infty(\Theta) p_0 \mathrm{d}\Theta + \int_0^{2\pi} \int_0^{\infty} T\left(v_0 + \alpha y \frac{\partial u_0}{\partial \Theta}\right) \mathrm{d}\Theta \mathrm{d}y}{\int_0^{2\pi} \int_0^{\infty} T \frac{\partial u_0}{\partial \Theta} \mathrm{d}\Theta \mathrm{d}y}. \tag{2.21}$$

At this stage we can write down the phase conservation equation correct to order $\delta^{1/3}$. We obtain

$$\frac{\partial \alpha}{\partial T} + \frac{\partial \Omega_0}{\partial \alpha} \frac{\partial \alpha}{\partial X} = -\delta^{1/3} \frac{\partial \Omega_1}{\partial X} + O(\delta^{2/3}), \tag{2.22}$$

and the expansion procedure given above can in principle be continued to any order. We postpone a discussion of the implications of (2.22) until we have described the results of the calculations required to determine $\alpha, \Omega_0, \Omega_1$.

## 3. The numerical work

The system (2.12) is periodic in $\Theta$ so we seek a solution by expanding for example $v_0$ in the form

$$v_0 = \sum_{-\infty}^{\infty} v_{0m}(y) \mathrm{e}^{im\Theta}. \tag{3.1}$$

After eliminating $p_{0m}$ and some linear terms proportional to $u_{0m}$ from the $\Theta$ momentum equation, the equation to determine $v_{0m}$ may be written in the form

$$\frac{\mathrm{d}^4 v_{0m}}{\mathrm{d}y^4} - im\{\alpha u_{00} - \Omega\} \frac{\mathrm{d}^2 v_{0m}}{\mathrm{d}y^2} = R_m, \tag{3.2}$$

where $R_m$ is a nonlinear function of $\{u_{om}\}$, $\{v_{0m}\}$. The equation for the mean part of $u_0$ is then written as

$$\frac{\mathrm{d}^2 u_{00}}{\mathrm{d}y^2} = \frac{\mathrm{d}I}{\mathrm{d}y}, \tag{3.3}$$

where $I$ is a nonlinear function of $\{u_{0m}\}$, $\{v_{0m}\}$. The functions $u_{00}, v_{om}, v_{0my}$ must vanish at the wall whilst $u_{00y}, v_{omyy}$ tend to zero for large $y$. The remaining condition relates

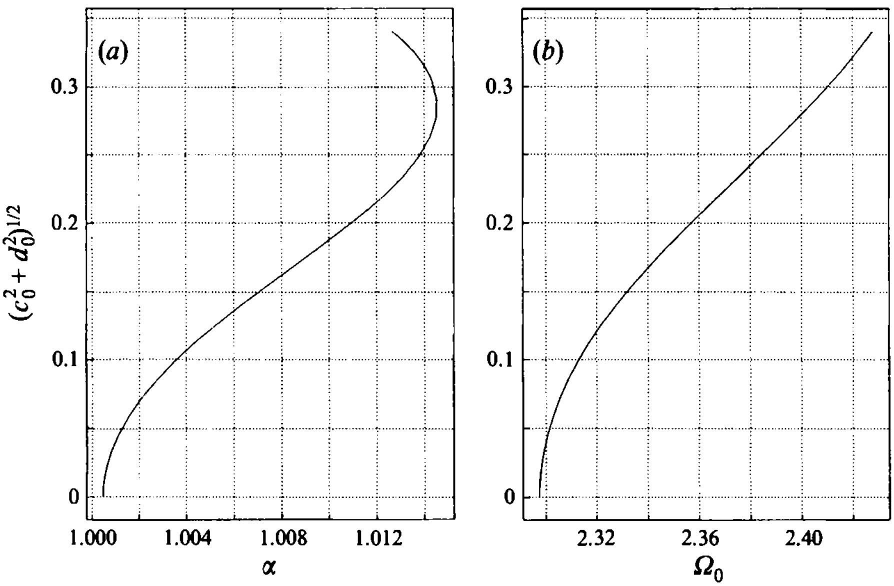


FIGURE 1. The dependence of (*a*) $\alpha$ and (*b*) $\Omega_0$ on $(c_0^2+d_0^2)^{1/2}$.

$v_{om}(w)$ and $v'''_{om}(0)$ using the equation of continuity. We used central differences to evaluate the derivatives in (3.2), (3.3) and a solution of the resulting nonlinear system was found by iteration after first restricting $m$ to be less than say $M$. In our iterative technique the right-hand sides of (3.2) and (3.3) were evaluated at the previous level of iteration and one boundary condition was replaced by

$$v_0'(\infty) = c_0 + \mathrm{i}d_0$$

where $c_0$ and $d_0$ are prescribed real constants. After the iterations converged we then adjusted $\alpha$ and $\Omega$ until the previously ignored boundary condition was satisfied. Note here that, because the solution of (2.12) is unique only up to a phase shift, the values of $\alpha$ and $\Omega$ obtained by this procedure are functions only of $\{c_0^2+d_0^2\}^{1/2}$.

The grid size and the value of $M$ were varied until converged results were obtained. In the following discussion the results presented correspond to $M = 32$, and 200 grid points in the $y$-direction with 'infinity' at $y = 10$. In figures 1(*a*) and 1(*b*) we show the dependence of $\alpha$ and $\Omega_0$ on the quantity $(c_0^2+d_0^2)^{1/2}$ which is a measure of the disturbance size. We see that, as predicted by Smith (1979*b*), finite-amplitude motion begins as a supercritical bifurcation from Blasius flow. In figure 2 we plot $\Omega_0$ as a function of $\alpha$. The calculations could not be continued beyond the point $F$ shown on the figure; we will return to this point later. We further note that $\Omega_0$ is a multiple-valued function of $\alpha$ for a range of values of $\alpha$ and that $\Omega_0'(\alpha)$ becomes infinite when $\alpha \sim 1.0145$. In figure 3 we show the shear stress as a function of $\Theta$ for a range values of $\Omega_0$. The results shown in this picture suggest a reason why figure 2 cannot be continued beyond the point $F$. We see that as the point $F$ is approached the shear stress approaches zero at a point. In figure 4 we show how the contribution to the shear stress from the different modes varies as $\Omega_0$ varies. We see that the higher modes grow rapidly as $F$ is approached. This suggests that the shear stress becomes singular as $F$ is approached.

Beyond the point $F$ our calculations failed to converge because the procedure used to solve (3.2)–(3.3) failed to drive the residuals below the tolerance level, $10^{-12}$, used

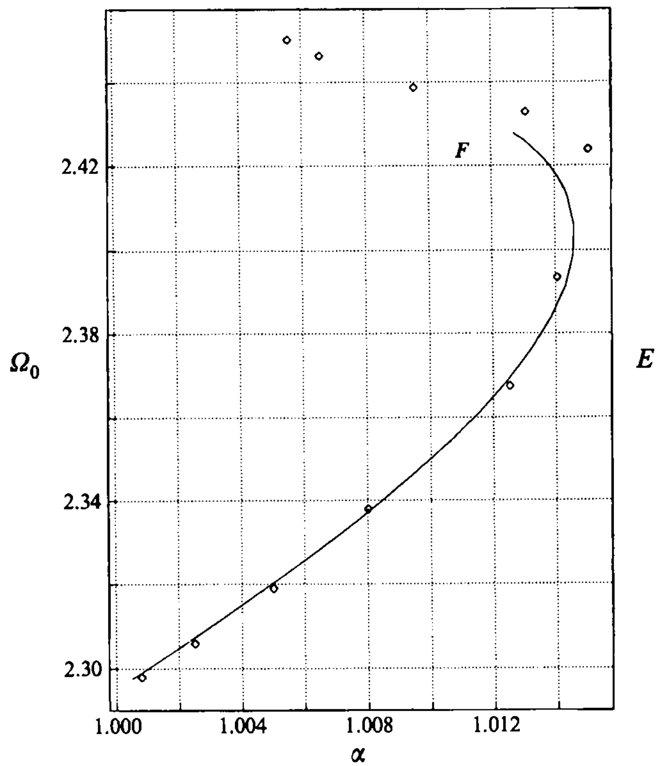


FIGURE 2. The dependence of $\Omega_0$ on $\alpha$. The symbols denote the results of Conlisk *et al.* (1987).

throughout our work. A similar result was found by Conlisk *et al.* (1987) who solved (2.12) by an indirect method. In their calculation the Tollmien–Sclichting waves were first forced by a wall motion and then their properties extrapolated as the forcing was reduced to zero. The results of Conlisk *et al.* (1987) have been plotted in figure 2 and we see that, on the whole, there is good agreement with our work when our program produced converged results. Some of the larger-amplitude results by Conlisk *et al.* were obtained by reducing the tolerance level in their iteration procedure. A similar reduction of the tolerance level in our code enables us to continue figure 2 for slightly larger disturbance amplitudes but we do not plot them for the following reasons. First, we found that a reduction of the tolerance level made our results very sensitive to the grid size. Second, a reduction of the tolerance level at best only enabled us to continue our calculation until $\alpha$ was reduced to about 1.012. Furthermore, the results of figure 3 suggest to us that the curve of figure 2 terminates at a point close to $F$ where all the harmonics are excited and a singularity has been encountered. Therefore it does not seem sensible to plot results obtained by reducing the tolerance level further. Further calculations were carried out at large frequencies in order to find finite-amplitude solutions of the type predicted by the Smith–Burggraf theory. Despite a careful search of the parameter regime identified by Smith & Burggraf (1985) no solutions could be found, but this does not mean that they do not exist.

The next calculation required concerned the constant $\Omega_1$ defined by (2.20). In order to calculate $\Omega_1$ from (2.20) it is necessary to compute the adjoint function $J = (M, N, S, T)$. In fact it is easier to solve the problem for $(u_1, v_1, p_1, A_1)$ directly and find the value of $\Omega_1$ which enables all the required boundary conditions to be

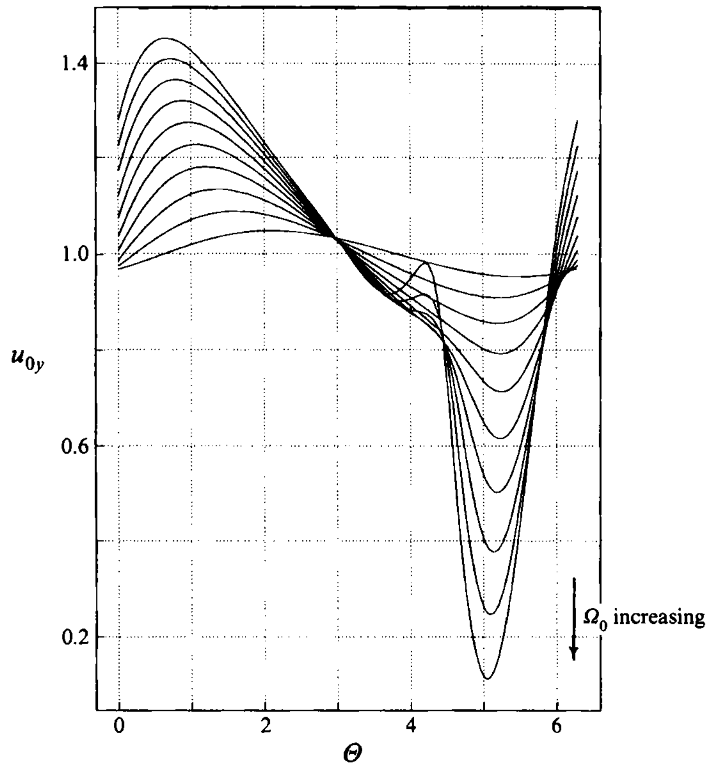


FIGURE 3. The shear stress as a function of $\Theta$ for $\Omega_0 = 2.2995$, 2.3041, 2.3125, 2.3245, 2.3398, 2.3575, 2.3763, 2.395, 2.4124, 2.4275.

satisfied. It was easier to compute $\Omega_1$ in this way because the system for $(u_1, v_1, p_1, A_1)$ can be solved using essentially the same iteration method as used above for the solution of (2.12). In figure 5(*a*) we show the dependence of $K$ on the wavenumber $\alpha$. The fact that $K$ is singular at $\alpha = \alpha_c = 1.0145$ is a direct consequence of the fact that $\Omega'(\alpha_c) = 0$. In figure 5(*b*) we show the dependence of $\bar{A}_0$ on $\alpha$ and we observe that $\bar{A}_{0\alpha}$ is respectively negative and positive on the lower and upper branches of figure 2. Here the upper and lower branches correspond to points on figure 2 which are respectively above or below $E$. The singularity in $\bar{A}_{0\alpha}$ is due to the fact that $\bar{A}_0$ continues to decrease when $\alpha$ passes through $\alpha_c$. The fact that both $K$ and $\bar{A}_{0\alpha}$ change sign at $\alpha_c$ means that viscous effects have essentially the same destabilizing role on the upper and lower branches when uniform wavetrains are considered; see the following section.

Now let us discuss the implications of our calculations for the evolution equation (2.22) which we recall determines the wavenumber $\alpha$ correct up to order $\delta^{1/3}$. The term on the right-hand side of (2.22) is due to viscous effects and the results of figure 5 imply that viscous effects are destabilizing. The zeroth-order approximation to (2.22) yields

$$\frac{\partial \alpha}{\partial T} + \omega_g \frac{\partial \alpha}{\partial X} = 0, \tag{3.4}$$

where $\omega_g$ is the group velocity. We see from figure 2 that the group velocity is negative for the upper branch and positive otherwise. This suggests that the upper branch solutions are physically irrelevant since their energy propagates upstream.

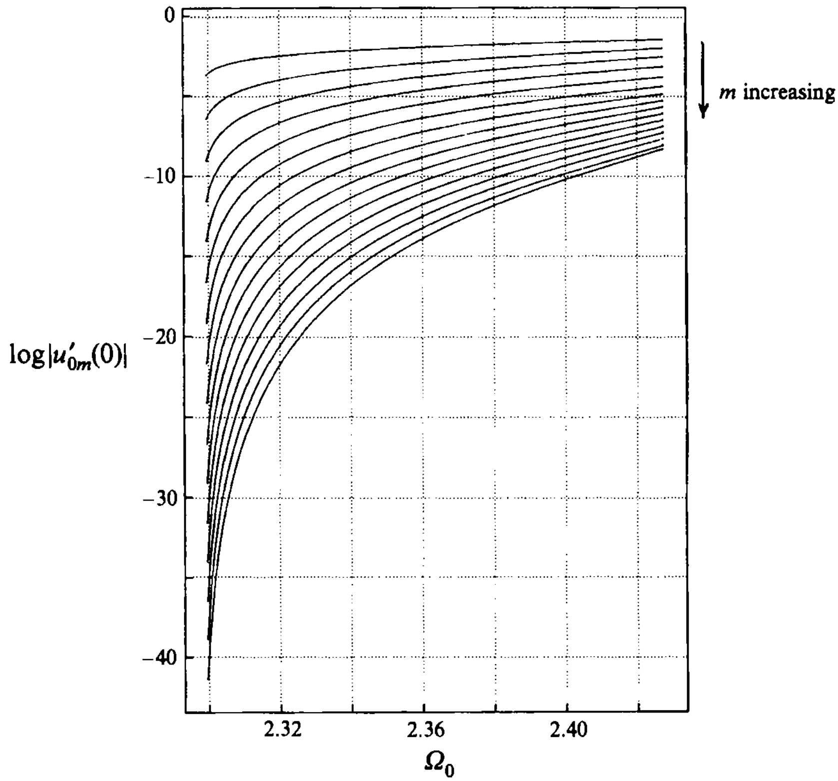


FIGURE 4. The shear stress $u'_{0m}(0)$ as a function of $\Omega_0$ for the first sixteeen modes.

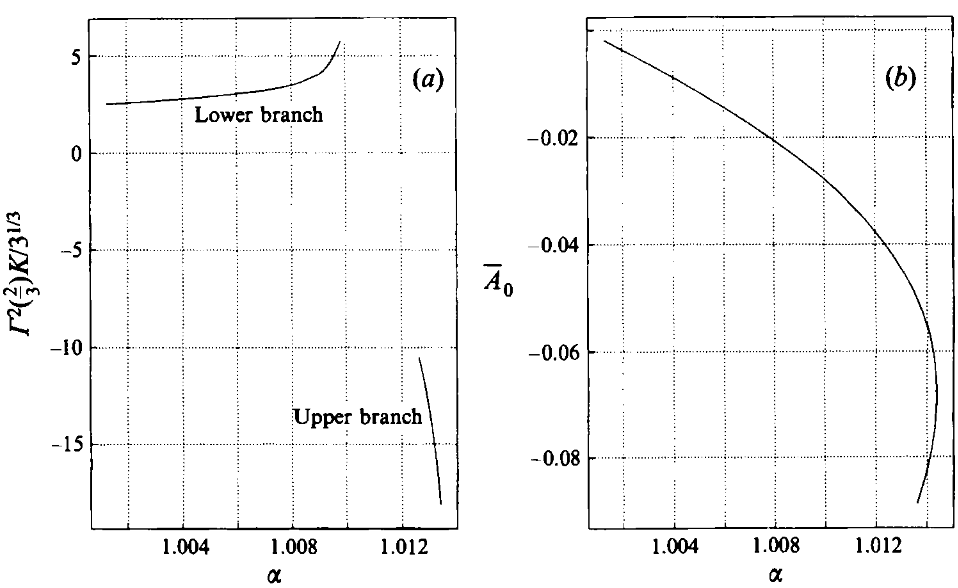


FIGURE 5. The dependence of (a) $K$ and (b) $\bar{A}_0$ on $\alpha$.

In fact, the form of figures 1(*b*) and 2, and the known result about the stability at small amplitude of Tollmien–Schlichting waves, Smith (1979*b*), suggests that solutions corresponding to the upper and lower branches would be found to be unstable and stable, respectively, if a Floquet analysis of them were carried out.

Suppose then that we consider the evolution of disturbances corresponding to the lower branch of figure 2. If at $T = 0$ we are given

$$\alpha = \bar{\alpha}(X),$$

then for $T$ positive we have

$$\alpha = \bar{\alpha}(X - \omega_g(\alpha)T),$$

which determines $\alpha$ implicitly since $\omega_g$ is a function of $\alpha$. It is well known (e.g. Whitham 1974), that for positive $\omega_g'$ the above solution will become multivalued after a finite time if the initial data has a compressive part. This suggests that finite-amplitude Tollmien–Schlichting waves will develop discontinuities in wavenumber and frequency as they propagate downstream. When such shocklike structures develop, (3.4) is no longer valid, and the viscous term must be brought into play. We might expect that the situation then is similar to that for Burgers equation (see Whitham 1974), where viscous effects smooth out shocklike solutions but do not prevent their development. Now we shall concentrate on a case where more analytical progress is possible and investigate nearly uniform wavetrains.

## 4. Uniform wavetrains and their stability

Suppose that $(\alpha_0, \Omega_0(\alpha_0))$ is some point on the curve shown in figure 2. The corresponding wave with

$$\Theta = \Theta(\alpha_0 X - \Omega_0(\alpha_0)T)$$

corresponds to a constant frequency/wavenumber solution of the full two-dimensional triple-deck problem for Tollmien–Schlichting waves. The stability of this system can be readily investigated by use of the phase equation (2.22). We first write $\alpha = \alpha_0 + \Delta$, where $\Delta$ is small, and then (2.19) becomes

$$\frac{\partial \Delta}{\partial T} + \Omega_0'(\alpha_0)\frac{\partial \Delta}{\partial X} + \Omega_0''(\alpha_0)\Delta\frac{\partial \Delta}{\partial X} = -\frac{3}{2\pi}\mathrm{Ai}(0)\chi^{4/3}\delta^{1/3}K(\alpha_0)\frac{\partial \bar{A}_0}{\partial \alpha}\frac{\partial}{\partial X}\int_{-\infty}^{\infty}\frac{\partial \Delta(q,T)}{\partial q}\beta(X-q)\mathrm{d}q + O(\delta^{1/3}\Delta^2, \Delta^3).$$

Note that $K\bar{A}_{0\alpha}$ is negative on both the lower and upper branches respectively of figure 2.

We can eliminate the term proportional $\Omega_0'(\alpha_0)$ by an appropriate Galilean transformation. If we then take $T = 0(\delta^{-1/3})$, $\Delta \sim \delta^{1/3}$ with $X = O(1)$ then, in the limit $\delta \to 0$, a suitably rescaled version of the above equation is

$$\frac{\partial \Lambda}{\partial \tau} + \Lambda\frac{\partial \Lambda}{\partial \xi} = \frac{\partial}{\partial \xi}\int_{-\infty}^{\infty}\frac{\partial \Lambda(q)}{\partial q}\beta(\xi - q)\mathrm{d}q. \tag{4.1}$$

Therefore the longwave instability of a uniform wavetrain of Tollmien–Schlichting waves is governed by the apparently new evolution equation (4.1).

Suppose now that at $\tau = 0$ there exists a small initial perturbation $\Lambda = \Lambda_0(\xi)$. The linearized form of (4.1) shows that $\Lambda$ evolves according to

$$\Lambda = \frac{1}{2\pi^{1/2}}\int_{-\infty}^{\infty}\Lambda_0^*(k)\exp[ik\xi + \Sigma\tau]\,\mathrm{d}k,$$

where $\Lambda_0^*$ is the transform of the initial data and

$$\Sigma(k) = \frac{-2\pi i k (ik)^{1/3}}{1 - \{i(3ik)^{2/3}(k+i0)^{1/2}(k-i0)^{1/2}\}\{\Gamma(\frac{1}{3})k^3\}}.$$

Here $(k \pm i0)$ are defined with branch cuts in the lower and upper half planes whilst the argument of $(ik)^{1/3}$ is $\pm\pi/6$ for $k$ on the positive and negative real axes respectively. The real part of $\Sigma$ is positive for real $k$. In fact the growth rate increases monotonically from zero when $|k|$ increases and behaves like $|k|^{4/3}$ at large wavenumbers. This means that a small disturbance to a uniform wavetrain of two-dimensional Tollmien–Schlichting waves always grows and the wavetrain is therefore modulationally unstable. Furthermore, since the growth rate tends to infinity in the short wavelength limit, linear theory certainly suggests that solutions to the evolution equation will develop a singularity after a finite time. It can be seen that the ultimate form of the disturbance depends on $\Lambda_0^*(k)$. More precisely we see that when $\Lambda_0(\xi)$ is sufficiently concentrated the solution will develop a singularity at a finite time. Thus for example an initial disturbance of Gaussian form will have $\Lambda_0^* \sim \exp(-k^2)$ and a bounded solution will occur for all time. However, if $\Lambda_0^* \sim \exp|k|^{4/3}$ the solution will become unbounded after a finite time. This is an important result because it says that a constant wavenumber/frequency solution of the two-dimensional triple-deck problem for Tollmien–Schlichting waves is always unstable to a long-wave instability. This is not uncommon in physical problems, e.g. the Stokes water wave, nonlinear optics. For a discussion of such modulational instabilities the reader is referred to Whitham (1974).

In fact the short wavelength instability of the linearized form of (4.1) suggests a rescaled version of the full nonlinear evolution equation when the amplitude of the disturbance becomes large. The new evolution equation is found to correspond to the case when the integral term in (4.1) is replaced by a simplified integral with transform obtained by replacing the denominator in the definition of $\Sigma$ by unity. We derive the simplified form of the evolution equation by assuming a new short lengthscale $O(\hat{\epsilon})$ in the $\xi$-direction and working out the asymptotic form of the integral term in (4.1) based on the assumption that $\Lambda$ now varies on this new short lengthscale. The amplitude of $\Gamma$ and time are then rescaled so that the terms on the left-hand side of (4.1) balance the integral term for small $\hat{\epsilon}$. Thus we write

$$\hat{\xi} = \frac{\xi}{\hat{\epsilon}}, \quad \hat{\tau} = \frac{\tau}{\hat{\epsilon}},$$

where $\hat{\epsilon}$ is small and seeking a solution of (4.1) with

$$\Lambda = \frac{\hat{\Lambda}(\hat{\xi}, \hat{\tau})}{\hat{\epsilon}^{1/3}},$$

The leading-order approximation to the evolution equation (4.1) in the limit of small $\hat{\epsilon}$ is obtained by noting that the dominant contribution to the integral in (4.1) occurs for positive values of the argument of the function $\beta$. Using the asymptotic expansion of the function $\beta(t)$ for small $t$ we obtain

$$\frac{\partial \hat{\Lambda}}{\partial \hat{\tau}} + \hat{\Lambda}\frac{\partial \hat{\Lambda}}{\partial \hat{\xi}} = \frac{\partial}{\partial \hat{\xi}} \int_{-\infty}^{\hat{\xi}} \frac{\partial \hat{\Lambda}(q)}{\partial q} (\hat{\xi} - q)^{-1/3} dq. \tag{4.2}$$

Note that in order to produce this canonical form for the evolution equation we have rescaled the variables by $O(1)$ amounts. The linearized form of (4.2) has exactly the same short wavelength instability as the original equation (4.1) so that if (4.2) is encountered after a localized breakdown of (4.1) it is likely that this new regime will itself suffer a subsequent breakdown. Later we shall present numerical results consistent with this suggestion. The breakdown of the linearized forms of (4.1), (4.2) is a direct consequence of the fact that the viscous-like operators on the right-hand sides of these equations are destabilizing. In (4.2) the term on the right hand side can be thought of as a fractional derivative of order $\frac{4}{3}$.

The discussion above suggests that an initial disturbance to either the linearized form of (4.1) or (4.2) will result in a finite time singularity if the transform of the initial disturbance does not decay faster than $e^{-|k|^{4/3}}$. However, the argument we have given completely ignores nonlinear effects. Unfortunately, nonlinear effects are unable to prevent this type of breakdown. If viscous effects are negligible, (4.1) and (4.2) reduce to the inviscid Burgers equation. It is well known, Whitham (1974), that Burgers equation develops a shock from rather arbitrary initial data. Therefore an initial disturbance which is not capable of causing a breakdown of the linear problem is modified by nonlinear effects until the locally rapidly varying structure associated with the generation of a shock is amplified by viscous effects into a singularity. We shall now present numerical results consistent with this description.

### 4.1. *Numerical solution of the wavenumber modulation equation*

Now let us discuss the results of some numerical investigations of the evolution equations (4.1) and (4.2). The calculations were carried out using a pseudospectral code kindly supplied to the author by D. Papageorgiu. The code was originally written in order to solve the Kuramot–Sivashinsky equation which is sufficiently similar to (4.1) and (4.2) for it to be trivially modified for the present investigation. The reader should consult for example Papageorgiu & Smyrlis (1991) for a full discussion of the code but for the sake of completeness we give the essential details here. The code uses a FFT approach to evaluate the nonlinear terms whilst the time integration is carried out using a predictor–corrector method. The code maintains spectral accuracy for initial data which is periodic in the spatial variable and the integral terms in the evolution equations conveniently transform into linear terms with wavenumber dependent coefficients in transform space. The results given in this section correspond to 4096 points in the spatial direction and a timestep $10^{-5}$. The results presented are grid-independent and are typical of those obtained for a wide vaiety of initial conditions. For brevity we report on results for just one initial condition.

In the first instance we integrated (4.1) subject to the initial condition

$$\Lambda = 2\pi \sin \xi, \tag{4.3}$$

and we note that the code preserves spectral accuracy for such a condition. Calculations with different initial conditions are of course possible but require more computational resources. Incidentally we note here in passing that each of the calculations reported on below took about two hours on a Silicon Graphics Indigo workstation.

In figure 6(*a*) we show the evolution of $\Lambda$ when $\tau$ increases. The initial steepening of the wave is due to the nonlinear term in the evolution equation. Later on in the calculation we see the development of a localized singularity just before the wave trough. Until the breakdown of the solution towards the end of the calculation the integral term is negligible and $\Lambda$ satisfies the inviscid Burgers equation. At the end of the calculation the steepening of the wave as a result of nonlinear effects produces a region where viscous effects come into play. At this stage a local singularity develops

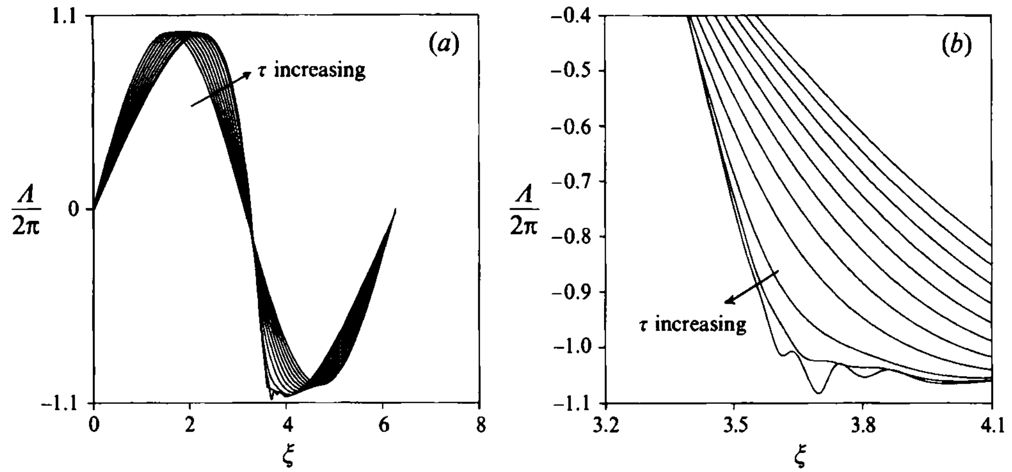


FIGURE 6. (*a,b*) Numerical results for the solution of (4.1) subject to the initial conditions (4.3). The curves shown correspond to $2\pi\tau = 0.012, 0.084, 0.156, 0.228, 0.3, 0.372, 0.444, 0.516, 0.588, 0.6612, 0.6264$.

and the calculations ultimately fail. The position where the breakdown occurs is of course a function of the initial disturbance and is independent of our discretization. In figure 6(*b*) we show the development of $A$ around the breakdown position.

In figure 7 the modulus of $A$ in wavenumber space is shown for increasing values of $\tau$. The energy in each wavenumber is seen to decay exponentially with wavenumber for most of the calculation. However, at later times we see that a kink develops in the curves and the exponential decay is lost. This change of structure is associated with the onset of the singularity. From this calculation it follows that a small perturbation to the wavenumber of a uniform wavetrain ultimately becomes unbounded and we conclude that the uniform wavetrain is modulationally unstable. The fate of the flow after the onset of the singularity is yet to be established but we note that 'in addition to terms of higher-order in $\delta$ neglected in the derivation of (2.22)' viscous diffusion in the streamwise direction or higher order terms in the Navier–Stokes equations might come into play and prevent the unbounded growth of the perturbation. An alternative outcome might be that the perturbation takes the flow back to a situation governed by triple-deck theory but without periodicity in the streamwise direction. We hope to report on this matter at a later stage.

Now let us discuss the results of our calculations involving the second wavenumber evolution equation (4.2). The equation was integrated subject to the same initial condition (4.3) and figures 8 and 9 show the evolution of $A$ in time. The results are essentially identical to those obtained for (4.1). Perhaps this is not surprising since the integral terms in the different evolution equations are virtually identical over most of wavenumber space. In the present case the steepening is again due to nonlinear effects and the breakdown occurs at a slightly earlier time. Otherwise the discussion given above for the results on the integration of (4.1) applies. We conclude that uniform wavetrains of two-dimensional Tollmien–Schlichting waves are unstable.

### 4.2. *Breakdown of the full nonlinear problem*

The calculations discussed above suggest that after a finite time a singular solution of (4.1) develops and that locally it is described by (4.2) in the first stages of the breakdown. Singular solutions of the linearized form of (4.2) are readily obtained

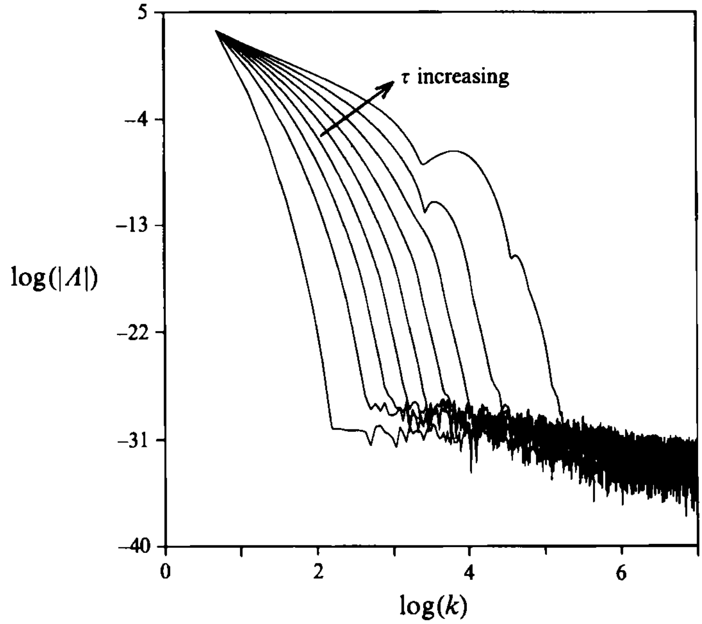


FIGURE 7. The modulus of $A(k)$ at times $2\pi\tau = 0.012,\ 0.084,\ 0.156,\ 0.228,\ 0.3,\ 0.372,\ 0.444,\ 0.516,\ 0.588$.

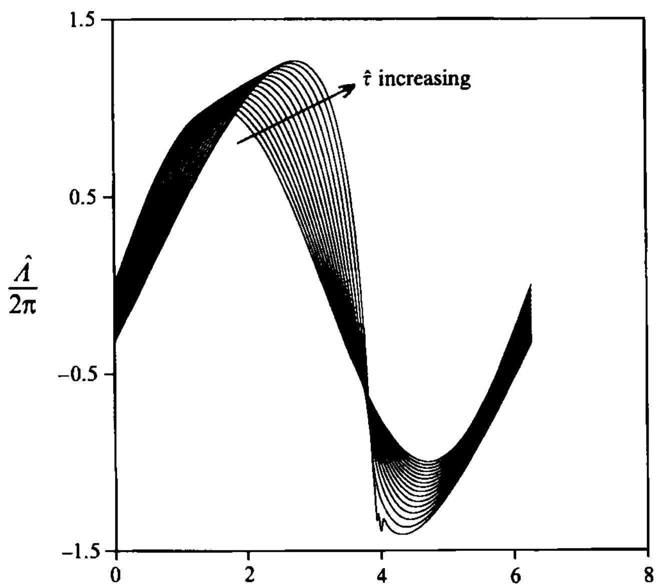


FIGURE 8. Numerical results for the solution of (4.2) subject to the initial conditions (4.3). The curves shown correspond to $2\pi\hat{\tau} = 0.012,\ 0.048,\ 0.84,\ 0.12,\ 0.156,\ 0.192,\ 0.228,\ 0.264,\ 0.3,\ 0.336,\ 0.372,\ 0.408,\ 0.444,\ 0.48,\ 0.516,\ 0.546$.

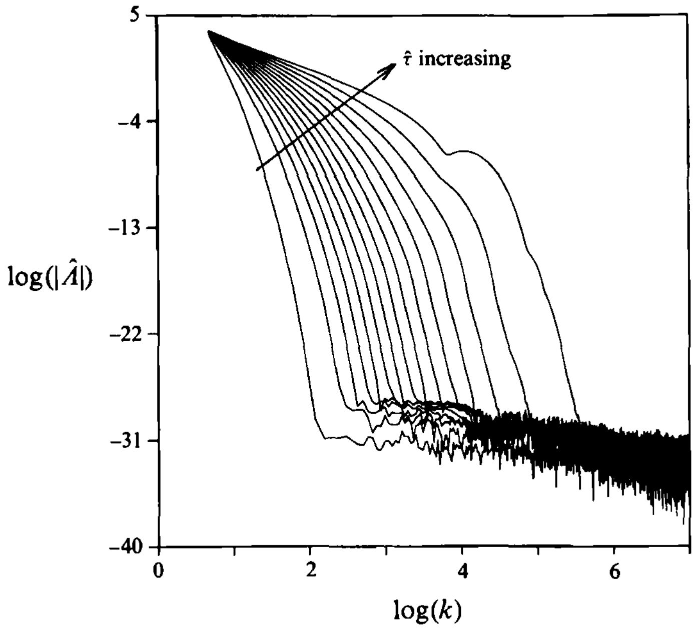


FIGURE 9. The modulus of $\hat{A}(k)$ at times $2\pi\hat{\tau} = 0.012$, 0.048, 0.84, 0.12, 0.156, 0.192, 0.228, 0.264, 0.3, 0.336, 0.372, 0.408, 0.444, 0.48, 0.516, 0.546.

using a Fourier transform technique. We do not present them here since they are presumably irrelevant in the nonlinear breakdown process.

Now let us consider a possible breakdown form for the full nonlinear system (4.1). If the breakdown is governed by the inviscid form of the equation then, following Brotherton-Ratcliffe & Smith (1987), we can, after a suitable shift of origin, write

$$A = |\tau|^{n-1}Q_0(\eta) + \cdots,$$

with $\eta = \xi/|\tau|^n$. The term on the right-hand side of (4.1) is then negligible for $n < \frac{3}{4}$ and $Q_0$ is then given implicitly by

$$\eta = -Q_0 - e_0 Q_0^3, \tag{4.4}$$

with $e_0$ a positive constant. However, (4.4) only determines $Q_0$ as a single-valued function of $\eta$ when $n = L/L - 1, L = 3, 5, 7, \cdots$, so that this type of structure is not possible. However, we can take $n = \frac{3}{4}$ in which case $Q_0$ satisfies

$$\tfrac{3}{4}\eta Q_0' + \tfrac{1}{4}Q_0 + Q_0Q_0' = \frac{\partial}{\partial \eta}\int_{-\infty}^{\eta} \frac{\partial Q_0}{\partial s}(s-\eta)^{-1/3}\mathrm{d}s. \tag{4.5}$$

The above integral equation must then be solved numerically.

## 5. The modulation equation for amplitude perturbations

We shall now use a multiple-scale approach to derive the equivalent of (4.1) directly from the two-dimensional triple-deck equations (2.1). Again the major effect of the modulation is to introduce a layer of depth $\delta^{-1/3}$ sitting on top of the lower deck. We again define slow variables $X$ and $T$ by writing

$$X = \delta x, \quad T = \delta t. \tag{5.1a, b}$$

It is then convenient to define $\hat{X}$, $\hat{Y}$ by

$$\hat{X} = X - \omega_g T, \quad \hat{T} = \delta^{1/3} T, \tag{5.2a, b}$$

where $\omega_g$ is a group velocity to be determined at higher order in our expansion procedure. Suppose then that we seek a solution of the triple-deck equations which is periodic in $\phi = \alpha x - \Omega t$ where $\alpha$ and $\Omega$ are now taken to be constant. In the lower deck we expand $u$ in the form

$$u = \sum_{n=0}^{\infty} \alpha^n \delta^{n/3} B^n(\hat{X}, \hat{T}) \frac{\partial^n u_0(\phi, y)}{\partial \phi^n} + \delta^{4/3} \left[ u_4(\phi, y, \hat{X}, \hat{T}) \right] + \delta^{5/3} u_5(\phi, y, \hat{X}, \hat{T}) + \cdots, \tag{5.3}$$

together with similar expansions for $v$ and $p$. We note at this stage that the summation term in (5.3) arises because of the translational invariance of a solution of the two-dimensional triple-deck equations. In addition we note that the first correction to the underlying mean state dependent on $\hat{X}$ arises at $O(\delta^{4/3})$ in (5.3). It should also be stressed at this stage that $u_0$, the 0(1) term in (5.3), is independent of the slow scales $\hat{X}$ and $\hat{T}$ and that $B(\hat{X}, \hat{T})$ is an amplitude function to be determined. Thus in (5.3) we identify the term $\delta^{1/3} B(\partial u_0/\partial \phi)$ as a small-amplitude perturbation to the periodic flow $u_0(\phi, y)$. The eigenfunction $\partial u_0/\partial \phi$ occurs because of the translational invariance of any $\phi$-periodic solution of the triple-deck problem. For our purposes here it is sufficient to consider the partial differential equations to determine $(u_4, v_4, p_4)$ and $(u_5, v_5, p_5)$. If the expansions for $u, v, p$ are substituted into the triple-deck equations, and the appropriate change of variables made, then we find that $(u_4, v_4, p_4)$ satisfies

$$\alpha \frac{\partial u_4}{\partial \phi} + \frac{\partial v_4}{\partial y} = -B_{\hat{X}} U_{0\phi}, \tag{5.4a}$$

$$\frac{\partial^2 u_4}{\partial y^2} - \alpha \frac{\partial p_4}{\partial \phi} - \alpha u_0 \frac{\partial u_4}{\partial \phi} - \alpha u_4 \frac{\partial u_0}{\partial \phi} - v_4 \frac{\partial u_0}{\partial y} - v_0 \frac{\partial u_4}{\partial y} = \left[ -\omega_g u_0 + \alpha u_0 u_{0\phi} \right] B_{\hat{X}}, \tag{5.4b}$$

$$p_{4y} = 0. \tag{5.4c}$$

These equations must be solved subject to $u_4 = v_4 = 0$, $y = 0$ whilst for large $y$ the appropriate conditions are

$$u_4 \to \hat{A}(\phi, \hat{X}, \hat{T}), \tag{5.5}$$

with

$$p_4 = \frac{\alpha}{\pi} \int_{-\infty}^{\infty} \frac{\partial \hat{A}/\partial s}{(\phi - s)} \mathrm{d}s. \tag{5.6}$$

Since the homogeneous form of the system for $(u_4, v_4, p_4)$ has the solution $(u_4, v_4, p_4) = (\partial/\partial \phi)\,(u_0, v_0, p_0)$ it follows that a solution exists only if an orthogonality condition is satisfied. This may be written down following the procedure used in §4. It is sufficient here to note that the condition determines the group velocity $\omega_g$ and that the expression obtained is identical to that derived in §3 after making a perturbation in the wavenumber. The solution of (5.4) is clearly of the form

$$(u_4, v_4, p_4) = B_{\hat{X}}(U_4, V_4, P_4), \hat{A} = B_{\hat{X}} \tilde{A},$$

where $(U_4, V_4, P_4)$ and $\tilde{A}$ are independent of $\hat{X}$, $\hat{T}$. The other main feature to appreciate about the solution of the $O(\delta^{4/3})$ problem is that at the edge of the

$y = O(1)$ region $U_4$ may be expressed in the form

$$U_4 \sim \tilde{A} = (A_M + A_F(\phi))B_{\hat{X}},$$

where $A_M$ is independent of $\phi$ and therefore corresponds to a mean flow correction. The reduction to zero of $A_M$ is achieved in the outer $O(\delta^{-1/3})$ region in a similar manner to that found in §3. We define the variable $\eta$ by

$$\eta = \delta^{1/3} y$$

and in the outer $\delta^{-1/3}$ layer $u$ is expanded in the form

$$u = \frac{\eta}{\delta^{1/3}} + \sum_{n=1}^{\infty} \alpha^n \delta^{n/3} B^n(\hat{X}, \hat{T}) \frac{\partial^n u_0}{\partial \phi^n}(\phi, \infty) + \delta^{4/3} U_M(\hat{X}, \eta) + \delta^{5/3} U_5 + \cdots,$$

where $U_M$ is the mean flow correction driven by $A_M$. The mean flow correction in the $\eta$ direction is $\delta^2 V_M$ and the linear problem to determine $(U_M, V_M, P_M)$ is identical to that with which we found $(u_M, v_M, p_M)$ satisfied in §2. Therefore it may be solved again using a Fourier transform technique, the solution is not repeated here. The mean flow at order $\delta^{4/3}$ then interacts with the $O(\delta^0)$ flow to produce an $O(\delta^{5/3})$ correction to the outer boundary condition for the disturbed flow in the $y = O(1)$ layer. Again the analysis follows closely that of §2 so we do not repeat it here, we find that $(u_5, v_5, p_5)$ must satisfy

$$\frac{\partial u_5}{\partial \phi} \to A_5 + \bar{\lambda} \frac{\partial}{\partial \hat{X}} \int_{-\infty}^{\hat{X}} \frac{(\partial B/\partial s)(s, \hat{T})}{(s - \hat{x})^{1/3}} \mathrm{d}s,$$

where $\bar{\lambda}$ is a constant. In the $y = O(1)$ region $(u_5, v_5, p_5)$ is found to satisfy (5.4*a,b*) but with the right-hand sides of these equations replaced by $[\cdot]$, and

$$[\cdot] + B_{\hat{T}} u_{0\phi} + B B_{\hat{X}} \left\{ v_4 \frac{\partial u_0}{\partial \phi} + u_0 \frac{\partial u_4}{\partial \phi} + v_4 \frac{\partial u_0}{\partial y} + v_0 \frac{\partial u_4}{\partial y} + \cdots \right\},$$

respectively. Here $[\cdot]$ denotes terms which contribute to the solvability condition which the system for $(u_5, v_5, p_5)$ must satisfy. The required condition is

$$\frac{\partial B}{\partial \hat{T}} + g_1 B \frac{\partial B}{\partial \hat{X}} = \hat{\lambda} \frac{\partial}{\partial \hat{X}} \int_{-\infty}^{\hat{X}} \frac{(\partial B/\partial s)(s, \hat{T})}{(s - \hat{X})^{1/3}} \mathrm{d}s, \tag{5.7}$$

where $g_1, \hat{\lambda}$ are constants and a suitable change of variables enables us to recover (4.1). However, as pointed out to the author by R. Hewitt, $g$, is identically $z_{oo}$ and (5.7) reduces to the linear form of (4.1). It can be shown also that all higher-order nonlinear terms do not contribute to (5.7).

## 6. The phase-equation approach for a parallel boundary layer

In §2 we derived the phase equation for Tollmien–Schlichting waves in a Blasius boundary layer. Such a boundary layer exists only at asymptotically large values of the Reynolds number and it was therefore appropriate to utilize the largeness of the Reynolds numbers in the description of the instability wave. However, linear descriptions of the evolution of Tollmien–Schlichting waves at finite Reynolds numbers have been given by, for example, Gaster (1974). Though such approaches do not give formal asymptotic approximations to the equations of motion it appears that they correctly predict the essential physics of the linear growth of Tollmien–Schlichting waves. Similarly, large-scale numerical simulations of nonlinear growth of Tollmien–Schlichting waves at finite Reynolds numbers have proved equally successful at reproducing experimental results, e.g. Wray & Hussaini (1984). Here we wish to investigate the phase-equation approach at finite Reynolds numbers but, in order to keep our asymptotic analysis formally correct, we choose to work with a parallel boundary layer which is an exact solution of the Navier–Stokes equations at all Reynolds numbers. We refer to the asymptotic suction boundary layer which has been investigated in the weakly nonlinear regime by Hocking (1975). Suppose then that the free-stream speed is $U_0$ and the suction velocity is $V_0$. We define a reference length $L = V/V_0$ and define the Reynolds number

$$R = \frac{U_0}{V_0}$$

but we assume that $R = O(1)$ in this section. Since we restrict ourselves to two-dimensional disturbances it is convenient for us to define a stream function $\psi$ and work with the vorticity equation in the form

$$\frac{\partial \nabla^2 \psi}{\partial t} + \frac{\partial(\nabla^2 \psi, \psi)}{\partial(x, y)} = \frac{1}{R}\nabla^4 \psi, \tag{6.1}$$

which is to be solved subject to

$$\psi_y = 0, \;\; \psi_x = 1, \;\; y = 0, \tag{6.2a}$$

$$\psi_y \to 1, \;\; y \to \infty \tag{6.2b}$$

In the absence of an instability the stream function $\psi$ is given by

$$\psi = \psi_0(x, y) = y + e^{-y} - x. \tag{6.3}$$

This flow is unstable to two-dimensional Tollmien–Schlichting waves for $R > 54370.0$ and the band of unstable wavenumbers tends to zero when $R \to \infty$. Here we assume $R$ is $O(1)$ and assume that an $O(1)$ amplitude wave system is superimposed on (6.3). At finite $R$ the mean flow driven by the wave system is confined to the boundary layer. Therefore no outer adjustment layer is required even when the wave system evolves slowly in the downstream direction, see Hocking (1975) for a discussion of this point. Suppose that $X, T$ are defined by

$$X = \delta x, \;\; T = \delta t,$$

then we define

$$\Theta = \frac{1}{\delta}\theta(X, T), \;\; \alpha = \theta_X, \;\; \Omega = -\Theta_T,$$

and expand $\Omega$ in the form

$$\Omega = \Omega_0 + \delta\Omega_1 + \cdots. \tag{6.4}$$

We seek a solution of (6.1) by writing

$$\psi = \psi_0(X, y, \Theta, T) + \delta\psi_1(X, y, \Theta, T) + \cdots$$

and the leading-order problem is determined by solving

$$-\Omega_0 \frac{\partial \nabla_1^2 \psi_0}{\partial \Theta} + \alpha \frac{\partial(\nabla_1^2 \psi_0, \psi_0)}{\partial(\Theta, y)} = \frac{1}{R}\nabla_1^4 \psi_0, \tag{6.5}$$

$$\psi_{0y} = 0,\ \alpha\psi_{0\Theta} = 1,\ y = 0,$$

$$\psi_{0y} \to 1,\ y \to \infty,$$

with

$$\nabla_1^2 \equiv \frac{\partial^2}{\partial y^2} + \alpha^2\frac{\partial^2}{\partial \Theta^2},$$

and periodicity in $\Theta$ for $\{\psi_0 - \Theta/\alpha\}$. The required frequency and wavenumber, $\Omega_0$ and $\alpha$, must be found numerically once some measure of the disturbance size is specified. We will not attempt such a calculation here but we note from the finite-Reynolds-number calculation of Hocking (1975) and the large-Reynolds-number theory of Smith (1979*a,b*) that both sub and supercritical bifurcations to finite amplitude Tollmien–Schlichting waves are possible. For the purposes of our discussion we simply assume that the nonlinear eigenrelation $\Omega_0 = \Omega_0(\alpha, R)$ is known. At order $\delta$ we find that $\psi_1$ is determined by

$$-\Omega_0\frac{\partial\nabla_1^2\psi_1}{\partial\Theta} - \alpha\frac{\partial(\psi_0,\nabla_1^2\psi_1)}{\partial(\Theta,y)} - \alpha\frac{\partial(\psi_1,\nabla_1^2\psi_0)}{\partial(\Theta,y)} - \frac{1}{R}\nabla_1^4\psi_1 = \Omega_1\frac{\partial\nabla_1^2\psi_0}{\partial\Theta} + \frac{\partial\alpha}{\partial X}M(\Theta,X,y). \quad (6.6)$$

where

$$M = M_L + M_N.$$

Here the linear and nonlinear functions $M_L$ and $M_N$ are defined by

$$M_L = R^{-1}\{4\alpha\nabla_1^2\psi_{0\Theta\alpha} + 4\alpha^2\psi_{0\Theta\Theta\Theta} + 2\nabla_1^2\psi_{0\Theta}\} + \Omega_0'(\alpha)\nabla_1^2\psi_0 + \Omega_0(2\alpha\psi_{0\alpha\Theta\Theta} + \psi_{0\Theta\Theta}),$$

and

$$M_N = -\psi_{0y}(\psi_{0\alpha yy} + 3\alpha\psi_{0\Theta\Theta} + 3\alpha^2\psi_{0\Theta\Theta\alpha}) + \alpha\psi_{0\Theta}(\psi_{0\Theta y} + 2\alpha\psi_{0\Theta y\alpha}) - \nabla_1^2\psi_{0y}\psi_{0\alpha}.$$

The system must be solved subject to periodicity in $\Theta$ whilst the boundary conditions in $y$ are

$$\psi_1 = \frac{\partial\psi_1}{\partial y} = 0,\ y = 0 \quad (6.7a)$$

$$\psi_1 \to q(X),\ y \to \infty. \quad (6.7b)$$

Here $q(X)$ represents a mean flow normal to the wall at infinity. This flow is essentially driven through the equation of continuity by the $O(1)$ streamwise velocity component. A solvability condition is required if (6.6)–(6.7) is to have a solution since the translational invariance of (6) means that $\psi_1 = \partial\psi_0/\partial\Theta$ is a solution of the homogeneous form of (6.6)–(6.7). It is worth pointing out at this stage that, if we were performing a calculation in a region of finite depth, then a pressure eigenfunction would have to be allowed for at leading order in order that $q$ should be reduced to zero. The requirement for such an eigenfunction is well known in weakly nonlinear stability theory; see for example Davey, Hocking & Stewartson (1974) or DiPrima & Stuart (1975). The solvability condition can be found by writing

$$\boldsymbol{Z} = (\psi, \alpha\psi_\Theta, \psi_y, \nabla_1^2\psi, \alpha\nabla_1^2\psi_\Theta, \nabla_1^2\psi_y)^T,$$

in which case the homogeneous form of (6.6) is

$$\alpha\frac{\partial}{\partial\Theta}\boldsymbol{AZ} + \frac{\partial}{\partial y}\boldsymbol{BZ} + \boldsymbol{CZ} = 0,$$

where $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$ are $6 \times 6$ matrices defined by

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\boldsymbol{B} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\boldsymbol{C} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -R\nabla_1^2\psi_{0y} & \alpha R\nabla_1^2\psi_{0\Theta} & 0 & R[\alpha\psi_{0y} - \Omega] & -R\alpha\psi_{0\Theta} \end{bmatrix}.$$

The system adjoint to that given above is

$$-\alpha\frac{\partial}{\partial\Theta}\boldsymbol{A}^T\boldsymbol{Q} - \frac{\partial}{\partial y}\boldsymbol{B}^T\boldsymbol{Q} + \boldsymbol{C}^T\boldsymbol{Q} = 0, \tag{6.8}$$

together with conditions of periodicity in $\Theta$ and

$$q_5 = q_6, \quad y = 0, \infty. \tag{6.9}$$

The condition that (6.6)–(6.7) has a solution then becomes

$$\Omega_1 = -\frac{\partial\alpha}{\partial X}f_1, \tag{6.10}$$

with

$$f_1 = \frac{\displaystyle\int_0^{2\pi/\alpha}\int_0^\infty M(\Theta, X, y)q_6 \mathrm{d}\Theta \mathrm{d}y}{\displaystyle\int_0^{2\pi/\alpha}\int_0^\infty \frac{\partial\nabla_1^2\psi_0}{\partial\Theta}q_6 \mathrm{d}\Theta \mathrm{d}y}.$$

The phase condition $\Omega_X + \alpha_T = 0$ correct up to $O(\delta)$ becomes

$$\alpha_T + \alpha_X\Omega_0(\alpha) = \frac{\partial}{\partial X}\left\{\alpha_X f_1(\alpha)\right\}\delta. \tag{6.11}$$

In order to examine the stability of a uniform wavestream solution we write

$$\alpha = \alpha_0 + \Delta,$$

with $\Delta << \alpha_0$, and (6.11) after a suitable change of scales becomes

$$\Lambda_\tau + \Lambda\Lambda_\xi = \pm\Lambda_{\xi\xi}. \tag{6.12}$$

Thus, we obtain the surprising result that the modulational instability of a two-dimensional wave system in a boundary layer at finite Reynolds numbers is governed by Burgers equation. In (6.12) the $\pm$ signs correspond to the cases when diffusion effects are stabilizing/destabilizing respectively. Without calculating $\Omega_1$ we do not know which sign is appropriate for the problem under consideration here so we will discuss both possibilities. However, we can simply quote the known results about Burgers equation for each case and the implications for the stability of a uniform wavetrain are essentially the same. A full discussion of results quoted below can be found in Whitham (1974) and Howard & Kopell (1977).

If the positive sign is taken the solution remains bounded for all time and indeed localized or periodic solutions of (6.12) tend to zero when $T$ increases. However, even in the diffusively stable case weak shock solutions of (6.12) can develop; a discussion of this possibility is given by Whitham (1974), Howard & Kopell (1977). Thus, for any given uniform wavetrain whose instability is governed by (6.12) with the positive sign, an initial disturbance can be found which does not decay to zero at large times. The uniform wavetrain is therefore modulationally unstable.

If the negative sign is taken in (6.12) viscous effects are destabilizing and in fact finite time singularities are developed from a broad range of initial conditions. Again it follows that the uniform wavetrain is unstable.

We conclude that at finite Reynolds numbers modulational effects will either cause a finite time singularity to develop if viscous effects are destabilizing, or cause shock discontinuities in wavenumber and frequency in the stable case. In either case uniform wavetrains of two-dimensional Tollmien–Sclichting waves at finite amplitude should not be observed according to our theory.

## 7. Conclusion

We have used a phase-equation approach to determine the evolution of Tollmien–Schlichting waves at large and finite Reynolds numbers. In the high-Reynolds-number case we found that finite-amplitude disturbances, periodic in time and space, apparently exist only for a small range of values of the wavenumber $\alpha$. The upper branch in figure 2 describes modes with negative group velocity and so they are therefore of no physical interest. The lower branch on the other hand corresponds to waves with positive group velocity and for these modes the rate of change of the group velocity with $\alpha$ is positive. This means that (2.22), the leading-order approximation to the phase equation, will develop discontinuities after a finite time for many initial disturbances. It might be anticipated that viscous effects, which appear on the right-hand side of (2.22), might smooth out such discontinuities. We cannot be sure if that is the case until a numerical investigation of (2.22) is carried out. However, in §4 we investigated the particular case of uniform wavetrains and found that there viscous effects were destabilizing and it seems likely that this is also the case for (2.22). In effect this means that periodic solutions of the triple-deck equations are modulationally unstable. A possible form for the structure of a singular solution of the breakdown of the nonlinear form of (4.1) was found. The structure found was based on the structure found by Brotherton-Ratcliffe & Smith (1987). The question of how the Navier–Stokes equations alter their large-Reynolds-number structure in order to remove the singularities of (4.1) also remains open.

At finite Reynolds numbers we found that the evolution equation for a periodic wavetrain satisfies Burgers equation. Without extensive calculations we cannot say whether the viscous term in (6.12) has a positive or negative sign. If it turns out to be negative then viscous effects are again destabilizing and finite time singularities will occur. If the sign is positive then viscous effects are stabilizing. However, (6.12) can be solved exactly by the Cole–Hopf transformation and it is known, Whitham (1974), that even in the stable case shocks will in general develop. Thus we conclude that at large or finite Reynolds numbers a uniform wavetrain of Tollmien–Schlichting waves will break down with a singularity or shock developing after a finite time. This casts some doubt on the validity of large-scale simulations of Tollmien–Schlichting waves using Fourier series expansions in the streamwise direction.

In view of the fact that our analysis has been restricted to the two-dimensional case it is possible that three-dimensional effects might prevent the above predictions from occurring in practice. Nevertheless, experimental observations where the Tollmien–Schlichting wave is driven by a wavemaker suggest that the first step in the transition process is the linear growth of two-dimensional Tollmien–Schlichting waves followed by nonlinear saturation and three-dimensional effects coming into play, see for example Klebanoff, Tidstrom & Sargent (1962).

Finally, we close with a brief outline of how (2.22) and (4.1) are modified by non-parallel flow effects. Non-parallel effects only enter a triple-deck description of TS waves parametrically except in particular cases. The calculation we have performed concerns the question of whether two-dimensional TS waves remain stable once they are established.

Within the framework of (2.1) non-parallel effects manifest themselves through the pressure displacement law which allows for elliptic effects in the streamwise direction. However, the slow spatial evolution of the unperturbed shear flow does not enter (2.1) since it can be scaled out of the problem. For the weakly nonlinear growth of Tollmien–Schlichting waves Hall & Smith (1984) showed that non-parallel effects are important when the disturbance amplitude is $O(R^{-7/32})$ and lead to an amplitude equation of the form

$$\frac{\mathrm{d}C}{\mathrm{d}X} = XC - (1+\mathrm{i}\bar{\delta})C|C|^2, \tag{7.1}$$

with $\bar{\delta}$ a real constant. Thus non-parallel effects lead to the term $XC$ in (7.1) thereby causing the increased linear exponential growth of a small disturbance as it evolves in $X$. Any small disturbance amplifying by this means eventually becomes nonlinear and for large $X$ has $|C|^2 \sim X$. Let us now show how related terms can be incorporated into (2.22) and (4.1).

We note that (2.11*a*) proceeds in powers of $\delta^{1/3}$ and that the Reynolds number has been effectively scaled out of the problem by our assumption that the Tollmien–Schlichting wave system is described by the triple-deck system (2.1).This assumption means that the analysis given so far in this paper is formally valid for $\delta$ large compared to any positive power of $\epsilon = R^{-1/8}$. In order to reveal the effect of boundary-layer growth we now relax this condition and see which new terms play a role in (2.11*a*). Clearly (2.11*a*) must include terms proportional to powers of $\epsilon$ because of the higher-order terms in the triple-deck expansion but in addition there will be a term proportional to $\epsilon^3\delta^{-1}X$ obtained by expanding the streamwise dependence of the unperturbed flow in a Taylor series in the streamwise direction. We therefore now expand the frequency in the form

$$\Omega = \Omega_0 + \delta^{1/3}\Omega_1 + \epsilon\Omega_2 + \epsilon^2\Omega_3 + \epsilon^3\delta^{-1}X\Omega_4 + O(\epsilon^3, \epsilon^6\delta^{-2}, \delta^{2/3}). \tag{7.2}$$

The ordering of the terms in the above expansion depends on the relationship between $\delta$ and $\epsilon$. The term proportional to $\Omega_4$ is the first one dependent on the non-parallel nature of the basic flow. The first significant distinguished limit arises when $\delta \sim \epsilon$ when the terms proportional to $\Omega_3, \Omega_4$ become comparable but still small compared to the $O(\delta^{2/3})$ term. The next significant stage is when $\delta$ decreases to $\epsilon^{9/5}$ in which case the nonparallel term and the $O(\delta^{2/3})$ term are comparable. However, the crucial stage arises when $\delta$ decreases to $\epsilon^{9/4}$ in which case the terms proportional to $\Omega_1, \Omega_4$ play an equal role and viscous and non-parallel effects are comparable. Thus if we write

$$\epsilon^3 = h_1 \delta^{4/3},$$

with $h_1$ an $O(1)$ constant then (2.2), correct to second order, now gives

$$\frac{\partial \alpha}{\partial T} + \frac{\partial \Omega_0}{\partial \alpha}\frac{\partial \alpha}{\partial X} = -\delta^{1/3}(\frac{\partial \Omega_1}{\partial X} + h_1\Omega_4 + h_1 X\frac{\partial \Omega_4}{\partial X}\cdots). \tag{7.3}$$

A further decrease in the size of $\delta$ means that the term proportional to $\Omega_1$ should be dropped and non-parallel effects dominate the right-hand side of the equation for $\alpha$. This completes our description of how the phase equation changes its structure when $\delta$ is decreased. Note here that when we decrease $\delta$ we are in effect moving further away from the initial location $x^* = \bar{x}^*$ so the above different orderings of the right-hand sides of (7.2) correspond to moving further downstream. Thus the final form has the right-hand side of the phase equation dominated by non-parallel effects.

A similar procedure can be used to determine the appropriate modifications to (4.1) which is the evolution equation for the perturbed wavenumber of an initially uniform wavetrain. Here the crucial scaling brings in non-parallel effects at the same stage as the integral term driven by viscous terms. The appropriate scaling now has

$$\epsilon^3 \delta^{-5/3} = h_2$$

with $h_2$ an $O(1)$ constant. Equation (4.1) then becomes

$$\frac{\partial A}{\partial \tau} + (A + h_3\tau)\frac{\partial A}{\partial \xi} = \frac{\partial}{\partial \xi}\int_{-\infty}^{\infty} \frac{\partial A(q)}{\partial q}\beta(\xi - q)\mathrm{d}q + h_4. \tag{7.4}$$

Here $h_3, h_4$ are constants proportional to $h_2$, and $h_4$ can be set equal to zero by a change of dependent variable. Therefore the non-parallel modulational equation for Tollmien–Schlichting waves in a growing boundary layer is (7.4) with $h_4 = 0$. The inviscid form of this equation can be solved and the solution develops in a manner similar to the solution of the same equation with $h_4 = 0$. Thus wave steepening occurs with a wide variety of initial data until viscous effects come into play and a singularity develops. Thus non-parallel effects do not alter the breakdown process associated with (4.1) and (4.2).

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